Rydberg Atomic Receiver

Lecture Notes on the Density-Matrix Description of Ladder EIT

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The Hamiltonian represents the energy of the system and the way in which the atomic states are coupled.

Step 2 – Hamiltonian in the Rotating Frame

In a ladder-EIT system, three atomic states are considered and denoted by $|1\rangle$, $|2\rangle$, and $|3\rangle$. The state $|1\rangle$ is the ground state, $|2\rangle$ is the intermediate excited state, and $|3\rangle$ is the upper state, for example a Rydberg state. The probe field couples the transition $|1\rangle \leftrightarrow |2\rangle$, while the control field couples the transition $|2\rangle \leftrightarrow |3\rangle$.

Under the rotating wave approximation and in the rotating frame, the effective Hamiltonian of the system can be written as

$$\hat{H} = -\hbar\Delta_p |2\rangle\langle 2| - \hbar\delta |3\rangle\langle 3| - \frac{\hbar}{2} \left[\Omega_p |2\rangle\langle 1| + \Omega_c |3\rangle\langle 2| + \text{h.c.}\right]. \tag{0-1}$$

In this expression, Δ_p is the detuning of the probe field from the transition $|1\rangle \leftrightarrow |2\rangle$:

$$\Delta_p = \omega_p - \omega_{21}. \tag{0-2}$$

The parameter δ is the two-photon detuning, which for a ladder system is usually defined as

$$\delta = \Delta_p + \Delta_c, \tag{0-3}$$

where Δ_c is the detuning of the control field from the transition $|2\rangle \leftrightarrow |3\rangle$:

$$\Delta_c = \omega_c - \omega_{32}. \tag{0-4}$$

The first term of the Hamiltonian,

$$-\hbar\Delta_p |2\rangle\langle 2|, \tag{0-5}$$

represents the effective energy of the intermediate state $|2\rangle$ in the rotating frame. If $\Delta_p = 0$, the probe field is exactly resonant with the transition $|1\rangle \rightarrow |2\rangle$. However, if $\Delta_p \neq 0$, the probe laser is detuned from resonance, and the state $|2\rangle$ appears shifted by this detuning in the rotating frame.

The second term,

$$-\hbar\delta |3\rangle\langle 3|, \tag{0-6}$$

represents the effective energy of the upper state $|3\rangle$. Since the state $|3\rangle$ is accessed through the absorption of two photons, namely one probe photon and one control photon, the important quantity for this state is the two-photon detuning δ. When $\delta = 0$, the total energy of the two photons exactly matches the energy difference between $|1\rangle$ and $|3\rangle$. This is the main condition for forming a dark state and observing EIT.

The terms inside the brackets represent the optical couplings:

$$\Omega_p |2\rangle\langle 1| \tag{0-7}$$

means that the probe field couples $|1\rangle$ to $|2\rangle$ with strength Ω_p. Here, Ω_p is the Rabi frequency of the probe field and determines the optical coupling strength of this transition.

Similarly,

$$\Omega_c |3\rangle\langle 2| \tag{0-8}$$

shows that the control field couples $|2\rangle$ to $|3\rangle$ with Rabi frequency Ω_c.

The term h.c. denotes the Hermitian conjugate. It adds the reverse coupling terms to the Hamiltonian:

$$\text{h.c.} = \Omega_p^* |1\rangle\langle 2| + \Omega_c^* |2\rangle\langle 3|. \tag{0-9}$$

Therefore, if the coupling part is fully expanded, we obtain

$$\Omega_p \left|2\right\rangle\!\left\langle 1\right| + \Omega_c \left|3\right\rangle\!\left\langle 2\right| + \Omega_p^* \left|1\right\rangle\!\left\langle 2\right| + \Omega_c^* \left|2\right\rangle\!\left\langle 3\right| . \tag{0-10}$$

The Hermitian-conjugate terms are necessary because physical couplings are bidirectional:

$$\left|1\right\rangle \leftrightarrow \left|2\right\rangle , \qquad \left|2\right\rangle \leftrightarrow \left|3\right\rangle . \tag{0-11}$$

The factor $\hbar/2$ in the coupling term comes from the standard definition of the Rabi frequency. In the dipole interaction between an atom and an electric field, the interaction energy has the form $-\mathbf{d}\cdot\mathbf{E}$. After applying the rotating wave approximation, the effective coupling usually appears with the factor $\hbar\Omega/2$.

Physical explanation: This Hamiltonian shows that, in the rotating frame, the intermediate state $|2\rangle$ is detuned from resonance by the probe detuning, while the upper state $|3\rangle$ is detuned by the two-photon detuning. On the other hand, the probe field connects $|1\rangle$ to $|2\rangle$, and the control field connects $|2\rangle$ to $|3\rangle$. When the two-photon detuning is close to zero, these couplings can create a dark state in which the lossy intermediate state $|2\rangle$ is not populated. As a result, probe absorption is suppressed and the EIT window appears.

0-1 Density-Matrix Description of a Ladder-EIT System

In a three-level ladder system, the three atomic states are defined as

$$\left|1\right\rangle = \text{ground state}, \qquad \left|2\right\rangle = \text{intermediate excited state}, \qquad \left|3\right\rangle = \text{Rydberg state}. \tag{0-12}$$

The first transition, $|1\rangle \leftrightarrow |2\rangle$, is coupled by the probe field with Rabi frequency Ω_p. The second transition, $|2\rangle \leftrightarrow |3\rangle$, is coupled by the control field with Rabi frequency Ω_c. Therefore, the overall structure of the system is

$$\left|1\right\rangle \xleftrightarrow[\Omega_p]{\text{probe}} \left|2\right\rangle \xleftrightarrow[\Omega_c]{\text{control}} \left|3\right\rangle . \tag{0-13}$$

Physical explanation: The probe field transfers the atom from the ground state to the intermediate state, while the control field connects the intermediate state to the Rydberg state. EIT appears when the two-photon pathway from $|1\rangle$ to $|3\rangle$ is established coherently.

0-1-1 Definition of Detunings

The detuning of the probe field from the transition $|1\rangle \leftrightarrow |2\rangle$ is defined as

$$\Delta_p = \omega_p - \omega_{21}. \tag{0-14}$$

The detuning of the control field from the transition $|2\rangle \leftrightarrow |3\rangle$ is

$$\Delta_c = \omega_c - \omega_{32}. \tag{0-15}$$

For a ladder system, the two-photon detuning is written as

$$\delta = \Delta_p + \Delta_c. \tag{0-16}$$

Physical explanation: The main condition for forming a dark state is that the two-photon detuning should be approximately zero. In other words, the total energy of the probe and control photons must match the energy difference between the ground state and the Rydberg state.

0-1-2 System Hamiltonian in the Rotating Frame

Under the rotating wave approximation, the Hamiltonian of the system can be written as

$$\hat{H} = -\hbar\Delta_p \left|2\right\rangle\!\left\langle 2\right| - \hbar\delta \left|3\right\rangle\!\left\langle 3\right| - \frac{\hbar}{2}\left[\Omega_p \left|2\right\rangle\!\left\langle 1\right| + \Omega_c \left|3\right\rangle\!\left\langle 2\right| + \text{h.c.}\right]. \tag{0-17}$$

Physical explanation: The first two terms describe the effect of the laser-frequency detunings from resonance. The terms containing Ω_p and Ω_c describe the coupling of the atomic states by the optical fields.

0-1-3 Density Matrix of the System

For a three-level system, the density matrix is written as

$$\hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix}. \tag{0-18}$$

In this matrix, the diagonal elements represent the state populations:

$$\rho_{11}, \quad \rho_{22}, \quad \rho_{33}. \tag{0-19}$$

The off-diagonal elements represent the quantum coherences between the states:

$$\rho_{21}, \quad \rho_{32}, \quad \rho_{31}. \tag{0-20}$$

In a ladder-EIT system, the most important elements are

$$\rho_{21} : \text{coherence associated with the probe-field response,} \tag{0-21}$$
$$\rho_{31} : \text{coherence between the ground state and the Rydberg state,} \tag{0-22}$$
$$\rho_{32} : \text{coherence associated with the control-field transition.} \tag{0-23}$$

Physical explanation: The element ρ_{21} determines how much the probe field is absorbed or transmitted. The element ρ_{31} is the key coherence between the ground state and the Rydberg state, and it is responsible for forming the dark state.

0-1-4 General Equation of Motion for the Density Matrix

The time evolution of the density matrix is described by the master equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}\right] + \mathcal{L}[\hat{\rho}]. \tag{0-24}$$

In this equation, the first term describes the coherent evolution caused by the Hamiltonian, while $\mathcal{L}[\hat{\rho}]$ accounts for decay and decoherence effects.

Physical explanation: The atom does not evolve only under the action of the lasers. The excited states also decay, and quantum coherences are destroyed because of interactions with the environment.

0-1-5 Population Equations

If the decay rate of the intermediate state $|2\rangle$ is Γ_2, and the decay rate of the Rydberg state $|3\rangle$ is Γ_3, the population equations are

$$\dot{\rho}_{11} = \Gamma_2 \rho_{22} + \frac{i}{2}\left(\Omega_p^* \rho_{21} - \Omega_p \rho_{12}\right), \tag{0-25}$$

$$\dot{\rho}_{22} = -\Gamma_2 \rho_{22} + \Gamma_3 \rho_{33} + \frac{i}{2}\left(\Omega_p \rho_{12} - \Omega_p^* \rho_{21}\right) + \frac{i}{2}\left(\Omega_c^* \rho_{32} - \Omega_c \rho_{23}\right), \tag{0-26}$$

$$\dot{\rho}_{33} = -\Gamma_3 \rho_{33} + \frac{i}{2}\left(\Omega_c \rho_{23} - \Omega_c^* \rho_{32}\right). \tag{0-27}$$

Physical explanation: The population of each level is redistributed by optical coupling and is also reduced or transferred to lower levels through decay.

0-1-6 Coherence Equations

For the coherence associated with the probe field, we have

$$\dot{\rho}_{21} = -\left(\gamma_{21} - i\Delta_p\right)\rho_{21} + \frac{i\Omega_p}{2}\left(\rho_{11} - \rho_{22}\right) + \frac{i\Omega_c^*}{2}\rho_{31}. \tag{0-28}$$

Physical explanation: This equation shows that the probe response does not depend only on the population difference between $|1\rangle$ and $|2\rangle$. Through ρ_{31}, the control field can also suppress the absorption of the probe field.

For the coherence between the ground state and the Rydberg state, we have

$$\dot{\rho}_{31} = -\left(\gamma_{31} - i\delta\right)\rho_{31} + \frac{i\Omega_c}{2}\rho_{21} - \frac{i\Omega_p}{2}\rho_{32}. \tag{0-29}$$

Physical explanation: The element ρ_{31} is the main dark-state coherence in ladder EIT. If this coherence is preserved, the atom can remain in a superposition of $|1\rangle$ and $|3\rangle$ without significantly populating the intermediate state $|2\rangle$.

For the coherence associated with the control field, we have

$$\dot{\rho}_{32} = -\left(\gamma_{32} - i\Delta_c\right)\rho_{32} + \frac{i\Omega_c}{2}\left(\rho_{22} - \rho_{33}\right) - \frac{i\Omega_p^*}{2}\rho_{31}. \tag{0-30}$$

Physical explanation: This coherence shows how the control field connects the intermediate state to the Rydberg state.

0-1-7 Weak-Probe Approximation

In many EIT experiments, the probe field is much weaker than the control field:

$$\Omega_p \ll \Omega_c. \tag{0-31}$$

In this regime, the probe field acts only as a measurement field and does not significantly change the atomic populations. Therefore, to zeroth order one can write

$$\rho_{11} \simeq 1, \qquad \rho_{22} \simeq 0, \qquad \rho_{33} \simeq 0. \tag{0-32}$$

Also, to first order in Ω_p, the effect of ρ_{32} can usually be neglected:

$$\rho_{32} \simeq 0. \tag{0-33}$$

Physical explanation: In the weak-probe regime, almost all atoms remain in the ground state, and the probe field only measures the optical response of the medium.

0-1-8 Steady-State Solution

In the steady state, we have

$$\dot{\rho}_{21} = 0, \qquad \dot{\rho}_{31} = 0. \tag{0-34}$$

Using the approximations $\rho_{11} \simeq 1$, $\rho_{22} \simeq 0$, and $\rho_{32} \simeq 0$, Eqs. (0-28) and (0-29) simplify to

$$0 = -\left(\gamma_{21} - i\Delta_p\right)\rho_{21} + \frac{i\Omega_p}{2} + \frac{i\Omega_c^*}{2}\rho_{31}, \tag{0-35}$$

$$0 = -\left(\gamma_{31} - i\delta\right)\rho_{31} + \frac{i\Omega_c}{2}\rho_{21}. \tag{0-36}$$

For compact notation, we define

$$\Gamma_{21} = \gamma_{21} - i\Delta_p, \qquad \Gamma_{31} = \gamma_{31} - i\delta. \tag{0-37}$$

Therefore,

$$0 = -\Gamma_{21}\rho_{21} + \frac{i\Omega_p}{2} + \frac{i\Omega_c^*}{2}\rho_{31}, \tag{0-38}$$

$$0 = -\Gamma_{31}\rho_{31} + \frac{i\Omega_c}{2}\rho_{21}. \tag{0-39}$$

From Eq. (0-39), we obtain

$$\rho_{31} = \frac{i\Omega_c}{2\Gamma_{31}}\rho_{21}. \tag{0-40}$$

Substituting Eq. (0-40) into Eq. (0-38), the probe-field response becomes

$$\rho_{21} = \frac{\dfrac{i\Omega_p}{2}\Gamma_{31}}{\Gamma_{21}\Gamma_{31} + \dfrac{|\Omega_c|^2}{4}}. \tag{0-41}$$

Physical explanation: The element ρ_{21} is the optical response of the medium to the probe field. The smaller ρ_{21} becomes, the weaker the probe absorption is, and the more clearly the EIT transparency window appears.

0-1-9 Optical Susceptibility of the Probe Field

The optical susceptibility associated with the probe field is proportional to the ratio of ρ_{21} to Ω_p:

$$\chi_p \propto \frac{\rho_{21}}{\Omega_p}. \tag{0-42}$$

Therefore,

$$\chi_p \propto \frac{i\Gamma_{31}}{\Gamma_{21}\Gamma_{31} + \dfrac{|\Omega_c|^2}{4}}. \tag{0-43}$$

The imaginary part of χ_p determines the absorption of the probe field, while the real part corresponds to dispersion and phase shift of the transmitted field:

$$\mathrm{Im}(\chi_p) \longrightarrow \text{absorption}, \qquad \mathrm{Re}(\chi_p) \longrightarrow \text{dispersion and phase shift}. \tag{0-44}$$

Physical explanation: The quantity χ_p specifies how strongly the atomic medium absorbs the probe field or changes its phase. In EIT, this quantity becomes very small at the center of the transparency window.

0-1-10 Ideal Condition for the Appearance of EIT

In the ideal case, the two-photon detuning is zero:

$$\delta = 0. \tag{0-45}$$

Also, if the decoherence between the ground state and the Rydberg state is negligible,

$$\gamma_{31} \simeq 0, \tag{0-46}$$

then

$$\Gamma_{31} = \gamma_{31} - i\delta \simeq 0. \tag{0-47}$$

As a result, from Eq. (0-41), we have

$$\rho_{21} \simeq 0. \tag{0-48}$$

Therefore,

$$\chi_p \simeq 0. \tag{0-49}$$

Thus, the absorption of the probe field almost disappears, and the transparency window appears.

Physical explanation: When the two-photon resonance condition is satisfied and the coherence between the ground state and the Rydberg state is not destroyed, the atom enters a dark state. In this state, the probe-absorption pathway through the intermediate state is removed, and the medium becomes transparent to the probe field.

0-1-11 Dark State in a Ladder-EIT System

In the ideal case, the dark state can be written as a superposition of the ground state and the Rydberg state:

$$|D\rangle = \frac{\Omega_c |1\rangle - \Omega_p |3\rangle}{\sqrt{|\Omega_c|^2 + |\Omega_p|^2}}. \tag{0-50}$$

In the regime $\Omega_p \ll \Omega_c$, this state is mostly similar to the ground state, but it also contains a small contribution from the Rydberg state:

$$|D\rangle \approx |1\rangle - \frac{\Omega_p}{\Omega_c} |3\rangle . \tag{0-51}$$

Physical explanation: The dark state is a combination of $|1\rangle$ and $|3\rangle$ with no contribution from the intermediate state $|2\rangle$. Since the intermediate state is the probe-light absorbing state, removing it from the dark state eliminates absorption and produces EIT.

0-1-12 Final Physical Result

In a ladder-EIT system, the control field creates coherence between the ground state and the Rydberg state. This coherence forms a dark state in which the lossy intermediate state $|2\rangle$ is not populated. Therefore, absorption of the probe field is suppressed, and a narrow transparency window appears in the transmission spectrum.

This mechanism is the basic principle of Rydberg atomic receivers. RF or microwave fields modify the Rydberg states and therefore change the EIT spectrum. This change can then be measured as a variation in the transmission of the probe light.

References